What is the Peak Temperature of a Square?

—When it is uniformly heated with fixed temperature on the border—

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Problem: A function $T(x, y)$ satisfies a Poisson's equation:

$$\nabla^2 T(x,y) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + 4 = 0, \tag{1}$$

for $|x| \leq 1/2$ and $|y| \leq 1/2$ with the boudary conditions

$$T(x, y = \pm\frac{1}{2}) = T(x = \pm\frac{1}{2}, y) = 0. \tag{2}$$

Calculate the value $T(0, 0)$.

Solution: The symmetries between x and y as well as x and $-x$ of the problem suggenst the following form of $T(x, y)$;

$$T(x,y) = -(x^2 + y^2) + a_0 + \sum_{n=1}^{\infty} a_n \left(\cos n\pi x \, \cosh n\pi y + \cosh n\pi x \, \cos n\pi y\right). \tag{3}$$

We can confirm that $T(x, y)$ certainly satisfies eq. (1).

The peak temperature at the origin $T(0, 0)$ is given by

$$T(0,0) = a_0 + 2\sum_{n=1}^{\infty} a_n, \tag{4}$$

and the coefficients a_j $(j = 0, 1, 2, \cdots)$ are to be determined from the boundary conditions, eqs. (2). The symmetries again enable us to consider only one of the four conditions, say,

$$T(x, y = \frac{1}{2}) = -x^2 - \frac{1}{4} + a_0 + \sum_{n=1}^{\infty} a_n \left(\cos n\pi x \, \cosh \frac{n\pi}{2} + \cosh n\pi x \, \cos \frac{n\pi}{2}\right). \tag{5}$$

The factor $\cosh n\pi/2$ in the right-hand-side of eq. (5) exponentially increases for large n, if $(x, y) \neq (0, 0)$. It means that the coefficient a_n exponentially decays. Therefore, we can expect that a small number of truncation of a_j with, say $j \leq 2$, gives fairly a good approximation of $T(0, 0)$. We set

$$T(x, y = \frac{1}{2}) \approx T^{(2)}(x), \tag{6}$$

where

$$\begin{aligned}
T^{(2)}(x) &= -x^2 - \frac{1}{4} + a_0 \\
&\quad + \sum_{n=1}^{2} a_n \left(\cos n\pi x \, \cosh \frac{n\pi}{2} + \cosh n\pi x \, \cos \frac{n\pi}{2} \right) \qquad (7) \\
&= -x^2 - \frac{1}{4} + a_0 \\
&\quad + a_1 \left(\cos \pi x \, \cosh \frac{\pi}{2} + \cosh \pi x \, \cos \frac{\pi}{2} \right) \\
&\quad + a_2 \left(\cos 2\pi x \, \cosh \pi + \cosh 2\pi x \, \cos \pi \right) \\
&= -x^2 - \frac{1}{4} + a_0 + a_1 \, \cosh \frac{\pi}{2} \, \cos \pi x \\
&\quad + a_2 \left(\cos 2\pi x \, \cosh \pi - \cosh 2\pi x \right) \qquad (8)
\end{aligned}$$

To fix a_0, a_1 and a_2, we impose three boundary conditions at $x = 0, 1/3$, and $1/2$ on this $y = 1/2$ line.

$$\begin{aligned}
T^{(2)}(0) &= -\frac{1}{4} + a_0 + a_1 \, \cosh \frac{\pi}{2} + a_2 \left(\cosh \pi - 1 \right) = 0, \qquad (9) \\
T^{(2)}(\frac{1}{2}) &= -\frac{1}{4} - \frac{1}{4} + a_0 - 2 \, a_2 \, \cosh \pi, \qquad (10) \\
T^{(2)}(1/3) &= -\frac{1}{9} - 1 + a_0 + \frac{a_1}{2} \, \cosh \frac{\pi}{2} + a_2 \left(-\frac{1}{2} \, \cosh \pi - \cosh \frac{2\pi}{3} \right) = 0. \qquad (11)
\end{aligned}$$

The linear equations (9) to (10) are easily solved as

$$\begin{aligned}
a_2 &= -\frac{1/36}{2 \, \cosh \frac{2\pi}{3} - 1}, \qquad (12) \\
a_0 &= \frac{1}{2} + 2 a_2 \cosh \pi, \qquad (13) \\
a_1 &= \frac{1}{4} - a_0 - a_2 \left(\cosh \pi - 1 \right) / \cosh \left(\pi/2 \right). \qquad (14)
\end{aligned}$$

that are numerically calculated:

$$\begin{aligned}
a_0 &= 0.5889048254279103 \qquad (15) \\
a_1 &= -0.15125367793243916 \qquad (16) \\
a_2 &= 0.0038347646559121653 \qquad (17) \\
& \qquad (18)
\end{aligned}$$

Note the expected rapid decay of a_j.

Taking n to 2 in the summation of eq. (4), we get

$$T(0,0) \approx 0.2940669988748564 \qquad (19)$$